QUADRATIC EQUATIONS

Polynomial: An expression of the form $f(x) = a_0x^2 + a_1 x + a_2$ where a_0, a_1, a_2 are real constants. *i.e.*, $a_0, a_1, a_2 \in \mathrm{R}$ and x is a variable is called a polynomial in x.

Coefficients: The constants a_0, a_1, a_2 are called coefficients of the polynomial $f(x)$.

Degree of a Polynomial: The degree of an equation involving one variable is given by the highest power of the variable in the equation after the equation has been reduced to the rational integral form.

An equation of the first degree is also called linear equation.

An equation of the second degree is also called a quadratic equation.

An equation of the third degree is also called a cubic equation.

Equation: If two different polynomials in the same variable x become equal for some values of x or a polynomial is equated to zero, then it is called an *equation*, *i.e.*, the polynomial $f(x) = a_0x^2 + a_1 x + a_2 = 0$, $a_0 \neq 0$ is an equation of degree 2.

Identity: An identity is a statement of equality between two algebraic expressions but is satisfied for all values of x. e.g., $(x-1)(x-2) \equiv x^2 - 3x + 2$ is satisfied for all values of x. The sign of identity is $\equiv$.

Quadratic Expression: $f(x) = ax^2 + bx + c$, $a \neq 0$; $a, b, c \in C$ is called a quadratic expression or function or polynomial of the second degree with complex coefficients.

If $a, b, c \in R$, then $f(x)$ is called a *real polynomial.*

Quadratic Equation: If the polynomial $f(x)$ when equated to zero is satisfied by one or more particular values of x, then $f(x) = 0$ is called an equation.

The general form of the quadratic equation in x is $ax^2 + bx + c = 0$, when $a \neq 0$. For example $f(x) = x^2 - 3x + 2 = 0$ is a quadratic equation as it is satisfied for $x = 2$ and 1 only.

Roots of an Equation: We have, $ax^2 + bx + c = 0$

or $\quad a^2 x^2 + abx + ac \quad = \quad 0$

$$\text{or} \qquad \left(ax + \frac{1}{2}b\right)^2 = \frac{1}{4}(b^2 - 4ac)$$

$$\text{or} \qquad ax + \frac{1}{2}b = \pm\frac{1}{2}\sqrt{(b^2 - 4ac)}$$

$$\therefore \qquad x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

If α and β be the roots of the equation, and $\alpha > \beta$, then

$$\alpha = \frac{-b + \sqrt{(b^2 - 4ac)}}{2a} \quad \text{and}$$

$$\beta = \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}$$

$$\text{Obviously} \qquad \sigma_1 = \alpha + \beta = \frac{-b}{a}$$

$$\sigma_2 = \alpha\beta = \frac{c}{a}$$

$$\alpha - \beta = \frac{\sqrt{(b^2 - 4ac)}}{a}$$

The quantity $b^2 - 4ac = \Delta$ is called the discriminant of the equation.

Type-I: In the equation of the type $ax^{2n} + bx^n + c = 0$, if x^n is put equal to y the equation becomes $ay^2 + by + c = 0$ which is a quadratic equation.

For example, solve $4^{1+x} + 4^{1-x} = 10$

Here, $4^1 \times 4^x + 4^1 \times 4^{-x} = 10$

or, $4 \times 4^x + \frac{4}{4^x} = 10$

or, $4 \times 4^{2x} + 4 - 10 \times 4^x = 0$

Putting $4^x = y$ we have

$4y^2 - 10y + 4 = 0$

or, $2y^2 - 5y + 2 = 0$

or, $(y - 2)(2y - 1) = 0$

either, $y = 2$ or $y = \frac{1}{2}$

either, $4^x = 2$ or $\frac{1}{2}$

either, $2^{2x} = 2^1$ or 2^{-1}

or, $2x = 1$ or -1

$$\therefore \qquad x = \frac{1}{2} \text{ or } \frac{-1}{2}$$

Type-II: $az + \frac{b}{z} = c$, when a, b, c are constants.

For example, solve $\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = \frac{13}{6}$

Here, $\sqrt{\frac{x}{1-x}}$ and $\sqrt{\frac{1-x}{x}}$ are reciprocal of each other.

Putting $\sqrt{\frac{x}{1-x}} = y$, the given equation becomes

$$y + \frac{1}{y} = \frac{13}{6} \text{ or } 6y^2 - 13y + 6 = 0$$

$$\therefore \qquad y = \frac{13 \pm \sqrt{169-144}}{12} = \frac{3}{2}; \frac{2}{3}$$

But $$y = \sqrt{\frac{x}{1-x}}$$

$$\therefore \qquad \sqrt{\frac{x}{1-x}} = \frac{3}{2} \text{ or } \sqrt{\frac{x}{1-x}} = \frac{2}{3}$$

On squaring, we get

$$\therefore \quad \frac{x}{1-x} = \frac{9}{4} \qquad \Bigg| \qquad \frac{x}{1-x} = \frac{4}{9}$$

$$\Rightarrow \quad x = \frac{9}{13} \qquad \Bigg| \qquad x = \frac{4}{13}$$

Hence, the solution is $\left(\frac{4}{13}, \frac{9}{13}\right)$

Type-III: Equation of the type $(x+a)(x+b)(x+c)(x+d)+k=0$ when sum of two of the quantities a, b, c, d is equal to the sum of the other two, can be solved as shown below:

$$(x + 1)(x + 2)(x + 3)(x + 4) + 1 = 0$$

$$\Rightarrow \quad [(x + 1)(x + 4)][(x + 2)(x + 3)] + 1 = 0$$

$$[\because 1 + 4 = 2 + 3]$$

$$\Rightarrow \quad (x^2 + 5x + 4)(x^2 + 5x + 6) + 1 = 0$$

Let $x^2 + 5x = y$, then

$$(y + 4)(y + 6) + 1 = 0$$

$$\Rightarrow \quad y^2 + 10y + 25 = 0$$

either $(y + 5)^2 = 0$ or $y = -5$

$$\therefore \quad x^2 + 5x = -5$$

or $x^2 + 5x + 5 = 0$

or $x = \dfrac{-5 \pm \sqrt{25-20}}{2} = \dfrac{-5 \pm \sqrt{5}}{2}$

Type-IV: Equation of the type $\sqrt{ax+b} = k$

or $\quad \sqrt{ax+b} + \sqrt{cx+d} = k$

or $\quad \sqrt{ax+b} + \sqrt{cx+d} = \sqrt{ex+f}$

Working Rules

(i) Square both sides.

(ii) Transpose so that the expression under radical sign is on one side.

(iii) Square both sides again and solve the resulting equation.

(iv) Test all the values of x and reject those values which do not satisfy the given equation.

For example, solve $\sqrt{2x+1} + \sqrt{3x+2} = \sqrt{5x+3}$

Here, squaring both sides,

$$(2x+1) + (3x+2) + 2\sqrt{(2x+1)(3x+2)} = 5x+3$$

$$\Rightarrow \quad 2\sqrt{(2x+1)(3x+2)} = 0$$

$$\Rightarrow \quad \sqrt{(2x+1)(3x+2)} = 0$$

Squaring again,

$$(2x + 1)(3x + 2) = 0$$

$$\therefore \qquad x = -\frac{1}{2}, -\frac{2}{3}$$

These two values of x satisfies the above example.

Nature of the roots of the quadratic equation: The most general quadratic equation is $ax^2 + bx + c = 0$. The nature of the roots of the given quadratic equation $ax^2 + bx + c = 0$, depends upon its discriminant, D or Δ, where $\Delta = b^2 - 4ac$, $\forall\ a, b, c \in \mathrm{R}$.

Case-I: When $b^2 - 4ac = 0$; the roots are real, rational and equal.

Case-II: When $b^2 - 4ac < 0$; the roots are unequal and imaginary i.e., the roots are complex and conjugate to each other.

Case-III: When $\Delta = b^2 - 4ac > 0$ but not a perfect square. In this case the roots are real, irrational and unequal.

Case-IV: When $\Delta = b^2 - 4ac$ is perfect square. In this case the $\sqrt{b^2 - 4ac}$ is real and rational. The roots are real, rational and unequal.

Relation Between Roots and Co-efficients of a Quadratic Equation $ax^2 + bx + c = 0$ are

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a},$$

$$\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Sum of the roots $= \alpha + \beta = \frac{-b}{a} = \frac{-\text{coeff. of } x}{\text{coeff. of } x^2}$

Product of the roots $= \alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coeff. of } x^2}$

Symmetric function of the roots: An expression involving α and β which remains unchanged by interchanging α and β is called a symmetric function of α and β.

For example, the functions

$\alpha+\beta, \alpha^2+\beta^2, \alpha^3+\beta^3, \alpha\beta$ are symmetric function of α and β. Following results should be carefully noted:

(i) $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

(ii) $(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$

(iii) $\alpha^2-\beta^2 = (\alpha+\beta)(\alpha-\beta) = (\alpha+\beta)\sqrt{(\alpha+\beta)^2-4\alpha\beta}$

(iv) $\alpha-\beta = \left[(\alpha+\beta)^2-4\alpha\beta\right]^{\frac{1}{2}}$

(v) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\,\alpha\beta\,(\alpha + \beta)$

$= (\alpha + \beta)\,(\alpha^2 - \alpha\beta + \beta^2)$

$= (\alpha + \beta)\,[(\alpha + \beta)^2 - 3\alpha\beta]$

(vi) $\alpha^3 - \beta^3 = (\alpha - \beta)\,(\alpha^2 + \alpha\beta + \beta^2)$

(vii) $\alpha^4 + \beta^4 = [(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2\alpha^2\beta^2$

(viii) $\alpha^4 - \beta^4 = (\alpha^2 - \beta^2)\,(\alpha^2 + \beta^2)$

$= (\alpha - \beta)\,(\alpha + \beta)\,(\alpha^2 + \beta^2)$

(ix) $\alpha^5 + \beta^5 = (\alpha^3 + \beta^3)\,(\alpha^2 + \beta^2) - \alpha^2\beta^2\,(\alpha + \beta)$

Some Important Properties:

(i) If α and β are roots of $f(x) = ax^2 + bx + c = 0$, then $f(x) = a\,(x - \alpha)\,(x - \beta)$.

(ii) The equation whose roots are α, β is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

(iii) ***One Common root:*** Two quadratic equations $f(x) = a_1x^2 + b_1\,x + c_1 = 0$ and $g(x) = a_2x^2 + b_2x + c_2 = 0$ has only one common root α if,

$a_1\alpha^2 + b_1\alpha + c_1 = 0$, $a_2\alpha^2 + b_2\alpha + c_2 = 0$.

Hence, by the method of cross-multiplication, we get, $\dfrac{\alpha^2}{b_1c_2 - b_2\,c_1} = \dfrac{\alpha}{c_1a_2 - c_2a_1} =$

$$\frac{1}{a_1b_2 - a_2b_1},\ (a_1b_2 - a_2b_1 \neq 0)$$

Thus, the required condition for one common root is $(c_1a_2 - c_2a_1)^2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1)$, and if $c_1, c_2 \neq 0$ (i.e., $\alpha \neq 0$), then we have $\alpha = \dfrac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}$ or $\dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$

(iv) ***Both Common Roots:*** Both roots of the equation $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ are common iff

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

(v) If α is a repeated root of $f(x) = 0$, then α is also a root of the equation $f'(x) = 0$.

(vi) If α is repeated common root of $f(x) = 0$ and $g(x) = 0$, then α is also a common root of the equations $f'(x) = 0$ and $g'(x) = 0$.

Polynomial Function of a Root: Let $f(x) = ax^2 + bx + c = 0$ has a root α and $g(x)$ be a polynomial in α of any degree, then $g(x)$ can be reduced to a polynomial of one degree in α.

If $q(\alpha)$ and $r(\alpha)$ be the quotient and remainder when $g(\alpha)$ is divided by $f(\alpha)$, then

$g(\alpha) = q(\alpha)\, f(\alpha) + r(\alpha),\ 0 \leq \deg r(\alpha) < 2.$

Thus, $g(\alpha) = r(\alpha)$ as $f(\alpha) = 0$

For example, Let $f(x) = 2x^2 - 3x - 1 = 0$, has a root α, then $f(\alpha) = 2\alpha^2 - 3\alpha - 1 = 0$.

Let $\quad g(\alpha) = 4\alpha^4 - 3\alpha^2 + \alpha - 1$,

then dividing $g(\alpha)$ by $f(\alpha)$, we get

$$g(\alpha) = (2\alpha^2 + 3\alpha + 4)(2\alpha^2 - 3\alpha - 1) + (16\alpha + 3)$$

Hence, $\quad g(\alpha) = 16\alpha + 3$ (as $2\alpha^2 - 3\alpha - 1 = 0$)

Resolution of $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ into two linear factors: First we consider the quadratic expression in $x, y : f(x, y)$

$$= ax^2 + 2hxy + by^2 + 2gx + 2fy + c \qquad \text{.....(i)}$$

On equating it to zero and considering it as quadratic equation in x, we get

$$f(x, y) = ax^2 + 2x(hy + g) + (by^2 + 2fy + c) = 0$$

After solving this, we get

$$x = \left\{\frac{-(hy+g) \pm \sqrt{\left[(hy+g)^2 - a\left(by^2 + 2fy + c\right)\right]}}{a}\right\}$$

or, $ax + hy + g = \underline{+} \sqrt{\left[(h^2 - ab)y^2 + 2y(hg - af) + (g^2 - ac)\right]}$

In order that $f(x, y)$ may be the product of two linear factors of the form $lx + my + n$, the quantity under the radical sign must be perfect square and the co-efficient of y^2 i.e., $h^2 - ab > 0$.

Hence, $(hg - af)^2 = (h^2 - ab)(g^2 - ac)$ which on dividing by a, gives $abc + fgh - af^2 - bg^2 - ch^2 = 0$, and $h^2 - ab > 0$. ..... (*i*)

If $h^2 - ab = 0$, then *eqn.* (*i*) becomes perfect square.

For the necessity of the condition $h^2 - ab > 0$, consider the quadratic

$$f(x, y) = 2x^2 + 4xy + 6y^2 - 8x + 12y + 33$$

which satisfy the condition (*i*) but is not resolvable into linear factors. Here, $h^2 - ab < 0$.

(*i*) The above condition in the determinant form can be expressed as

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \text{ and } h^2 - ab \geq 0.$$

(*ii*) This result is very useful in analytical geometry to examine whether general equation of second degree in x, y represents two real straight lines or not.

Solutions of Some quadratic and Higher Degree Inequation:

1. For $a > 0$, $x^2 < a^2$ or $|x| < a$ iff. $-a < x < a$.
2. For $a > 0$, $x^2 > a^2$ or $|x| > a$ iff. $x < -a$ or $x > a$
3. For $a < \beta$, $(x - \alpha)(x - \beta) < 0$ iff $\alpha < x < \beta$.
4. For $a < \beta$, $(x - \alpha)(x - \beta) > 0$, iff. $x < \alpha$ or $x > \beta$.

For example, consider $3x^2 + 2x - 1 < 0$.

The inequation is equivalent to $x^2 + \frac{3}{2}x - \frac{1}{3} < 0$.

$$\text{i.e., } \left(x + \frac{1}{3}\right)^2 - 4/9 < 0$$

By using eqn. (*i*), we get

$$-\frac{2}{3} < x + \frac{1}{3} < \frac{2}{3}, \; i.e., -1 < x < \frac{1}{3}$$

Note: The above inequations can also be solved in the following way:

Consider $\quad f(x) = (x - \alpha)(x - \beta), \alpha < \beta$

$\quad f(x) = 0$ gives $x = \alpha, \beta$, then

	$x - \alpha$	$x - \beta$	$f(x)$
$x < \alpha$	$-$	$-$	$+$
$\alpha < x < \beta$	$+$	$-$	$-$
$x > \beta$	$+$	$+$	$+$

so $\quad f(x) = (x-\alpha)(x-\beta) < 0$ for $a < x < \beta$,

and $\quad f(x) = (x-\alpha)(x-\beta) > 0$ for $x < \alpha$ or $x > \beta$

i.e.,

$f(x)$: $\quad + \quad \alpha - \beta \quad +$

The above mentioned procedure can also be used in solving higher degree inequations and rational integral inequations. The procedure involve 3 steps.

Step I: Find x by putting numerator and denominator equal to zero and convert the function into linear factors. These values of x may be termed as critical points.

Step II: Put all these points on the number line in order.

Step III: Start with + sign from extreme right and then take alternately –ve and +ve signs.

$+ \quad - \quad + \quad - \quad +$

Then write the answer accordingly.

For example, (1) solve $(2x+1)(x-3)(x-1) > 0$

Here, $\quad$ Let $f(x) = (2x+1)(x-3)(x-1)$

$$f(x) = 0 \text{ gives } x = -\tfrac{1}{2}, 3, 1$$

Putting those roots in order, we get $x = -\frac{1}{2}, 1, 3$.

$f(x)$: $\quad - \quad \frac{1}{2} \quad + \quad 1 \quad - \quad 3 \quad +$

So $f(x) > 0$ in $-\frac{1}{2} < x < 1$ and $x > 3$

(2) Solve $f(x) = \dfrac{(x-1)(x+2)}{(x-5)} \leq 0$

Here critical points are 1, –2, 5.

$$\begin{array}{cccccccc} & & \times & & \times & & \times & \\ f(x): & - & -2 & + & 1 & - & 5 & + \end{array}$$

and $f(x) = 0$ for $x = 1$ and -2, $x \neq 5$ as $(x - 5)$ is in denominator.

So, $f(x) \leq 0$ for $x \leq -2$ and $1 \leq x < 5$

This can also be expressed as solution set

$$x \in [-\infty, -2] \cup (1, 5)$$

Sign of Quadratic Expression $f(x)$: $ax^2 + bx + c$.

Consider $f(x) = ax^2 + bx + c$, $a \neq 0$, a, b, $c \in R$

Case I: $\Delta = b^2 - 4ac < 0$, i.e., $f(x) = 0$ has imaginary roots.

$$f(x) = a\left\{x^2 + \frac{b}{a}x + \frac{c}{a}\right\}$$

$$= a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}$$

as $\Delta < 0$, $\dfrac{4ac - b^2}{4a^2} > 0$

$$\text{so } \left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\} > 0.$$

Thus $f(x) > 0$ for all $x \in R$ if $a > 0$, $f(x) < 0$ for all $x \in R$ if $a < 0$, *i.e.*, $f(x)$ and a have same sign for all $x \in R$.

Case II: $\Delta = 0$ then $f(x) = a\,(x + b/2a)^2$.
As $(x + b/2a)^2 \geq 0$ for all $x \in R$.
$\therefore$ We have $f(x) \geq 0$ for all $x \in R$ if $a > 0$.
$f(x) < 0$ for all $x \in R$ if $a < 0$.

Note: From case I and II, it is clear that
(i) $f(x) \geq 0$ for all $x \in R$ if $\Delta \leq 0$ and $a > 0$ and
(ii) $f(x) < 0$ for all real $x \in R$ if $\Delta \leq 0$; and $a < 0$.

For example, find a if $2x^2 + ax + 5 \geq 0$ for all real x.
Here, from the above discussion, it is clear that $\Delta \leq 0$ as co-efficient of $x^2 = 2 > 0$.

$$\therefore\ a^2 - 40 \leq 0,\ a^2 \leq \left(2\sqrt{10}\right)^2$$

$$\therefore -2\sqrt{10} \leq a \leq 2\sqrt{10}$$

Case III: $\Delta > 0$. In this case roots are real and distinct. Let α, β be the roots and $\alpha < \beta$. Then $f(x) = a\,(x - \alpha)\,(x - \beta)$.

If $a > 0$, then

$f(x)$: $>$ 0; α < 0, $\beta > 0$

i.e., if $a < x < \beta$; $f(x) < 0$,
if $x < \alpha$ or $x > \beta$; $f(x) > 0$.

and if $x < 0$, then

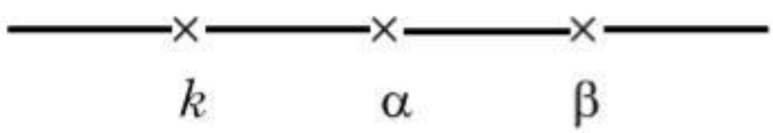
$f(x) : < 0; \; \alpha > 0, \; \beta < 0$

Hence, from the above discussion, it is clear that a and $f(x)$ have the same sign except when roots are real and x is lying between them.

Location of roots (Interval in which roots lie)

Let $f(x) = ax^2 + bx + c = 0$, $a, b, c \in \mathrm{R}$, $a > 0$ and α, β be the roots.

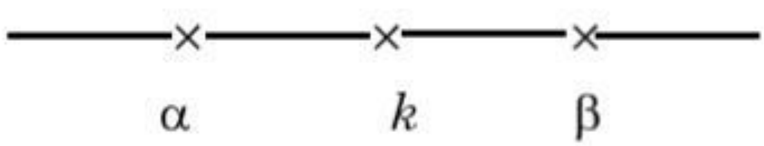
(*i*) Both roots of $f(x) = 0$ are greater than some fixed number k.



Roots must be real and as k does not lie between α and β. So $\Delta \geq 0$, $f(x) > 0$, also $\alpha + \beta > 2k$, *i.e.*, $\Delta \geq 0$, $f(x) > 0$, $-(b/a) > 2k$.

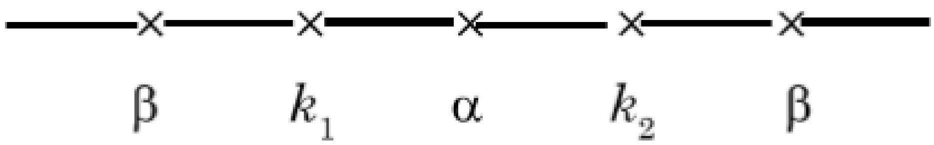
It should be noted that, both roots are less than K, if $\Delta \geq 0$, $f(k) > 0$, $-(b/a) < 2k$

(*ii*) One root is less than k and other is greater than k.



Here roots must be real and distinct so $\Delta > 0$, $f(k) < 0$.

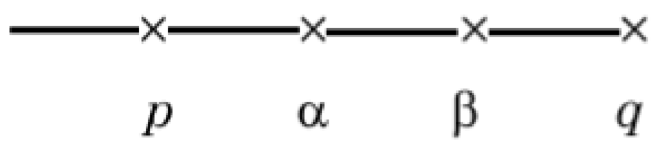
(*iii*) Exactly one root lies in the interval (k_1, k_2) *i.e.*,



$$\beta < k_1 < \alpha < k_2 \text{ or } k_1 < \alpha < k_2 < \beta,$$

then, one of $f(k_1)$ or $f(k_2)$ is negative and other is positive so $\Delta > 0$, $f(k_1)\, f(k_2) < 0$.

(*iv*) If both roots are confined between the numbers p, q,



Here, roots must be real so $\Delta \geq 0$, $f(p) > 0$, $f(q) > 0$,

$$p < \frac{\alpha + \beta}{2} < q, \text{ i.e., } p < -\frac{b}{2a} < q$$

(*v*) If $f(p)$ and $f(q)$ have opposite sign, $f(x) = 0$ must have at least one root between p and q *i.e.*, there exists at least one α such that

$p < \alpha < \text{q}$, $f(x) = 0$.

This result is also applicable for any polynomial equation of any degree and is due to *Rolle*.

(*vi*) If $f(x)$ be any polynomial such that $f(a) = f(b) = 0$, then $f'(x) = 0$ has at least one root between a and b *i.e.*, there exists at least one c such that

$$a < c < b \text{ and } f'(c) = 0$$

This result is a particular version for polynomial equations of a more general theorem calld Rolle's theorem.

Equation of higher degree: The equation $f(x) = a_0x^n + a_1 x^{n-1} + a_2 x^{n-2} + ...+ a_{n-1} x + a_n = 0$. $a_0, a_1 ..., a_n \in$ C (or R), $a_0 \neq 0$ is a polynomial equation of degree n. It has n and only n roots. Let $\alpha_1, \alpha_2 ....., \alpha_n$ be n roots, then

$$\sigma_1 = \Sigma\, \alpha_1 = -\frac{a_1}{a_0}, \sigma_2 = \Sigma\, \alpha_1\, \alpha_2 = \frac{a_2}{a_0} \text{ etc.}$$

In general $\sigma_r = \Sigma\alpha_1 ..... \alpha_r = (-1)^r \dfrac{a_r}{a_0}$

In particular, if α, β, γ be roots of

$$ax^3 + bx^2 + cx + d = 0,$$

We have

$$\begin{aligned}\sigma_1 &= \alpha + \beta + \gamma = -b/a \\ \sigma_2 &= \alpha\beta + \beta\gamma + \gamma\alpha = c/a \\ \sigma_3 &= \alpha\beta\gamma = (-d/a).\end{aligned}$$